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## Second order Nonlinear Difference Equations whose Eigenvalues are 1

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### 1 Introduction

At first we consider the following second order nonlinear difference equation,

$$\begin{cases} u(t+1) = U(u(t), v(t)), \\ v(t+1) = V(u(t), v(t)), \end{cases} \quad (1.1)$$

where  $U(u, v)$  and  $V(u, v)$  are entire functions for  $u$  and  $v$ . We suppose that the equation (1.1) admits an equilibrium point  $(u^*, v^*) = (0, 0)$ . Furthermore we suppose that  $U$  and  $V$  are written in the following form

$$\begin{pmatrix} u(t+1) \\ v(t+1) \end{pmatrix} = M \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + \begin{pmatrix} U_1(u(t), v(t)) \\ V_1(u(t), v(t)) \end{pmatrix},$$

where  $U_1(u, v)$  and  $V_1(u, v)$  are higher order terms of  $u$  and  $v$ . Let  $\lambda_1, \lambda_2$  be characteristic values of matrix  $M$ . For some regular matrix  $P$  which decided by  $M$ , put  $\begin{pmatrix} u \\ v \end{pmatrix} = P \begin{pmatrix} x \\ y \end{pmatrix}$ , then we can transform the system (1.1) into the following simultaneous system of first order difference equations (1.2):

$$\begin{cases} x(t+1) = X(x(t), y(t)), \\ y(t+1) = Y(x(t), y(t)), \end{cases} \quad (1.2)$$

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where  $X(x, y)$  and  $Y(x, y)$  are supposed to be holomorphic and expanded in a neighborhood of  $(0, 0)$  in the form

$$\begin{cases} X(x, y) = \lambda_1 x + \sum_{i+j \geq 2} c_{ij} x^i y^j = \lambda_1 x + X_1(x, y), \\ Y(x, y) = \lambda_2 y + \sum_{i+j \geq 2} d_{ij} x^i y^j = \lambda_2 y + Y_1(x, y), \end{cases} \quad (1.3)$$

or

$$\begin{cases} X(x, y) = \lambda x + y + \sum_{i+j \geq 2} c'_{ij} x^i y^j = \lambda x + X'_1(x, y), \\ Y(x, y) = \lambda y + \sum_{i+j \geq 2} d'_{ij} x^i y^j = \lambda y + Y'_1(x, y), \end{cases} \quad (1.4)$$

where  $\lambda = \lambda_1 = \lambda_2$ .

In this note we consider analytic solutions of difference system (1.2), making use of Theorems in [1] and [4]. We will seek an analytic solution of (1.2) under the conditions  $\lambda_1 = \lambda_2 = 1$  and definition (1.3). Further we suppose that

$$\begin{cases} X(x, y) = x + \sum_{i+j \geq 2, i \geq 1} c_{ij} x^i y^j = x + X_1(x, y), \\ Y(x, y) = y + \sum_{i+j \geq 2, j \geq 1} d_{ij} x^i y^j = y + Y_1(x, y), \end{cases} \quad (1.5)$$

where  $X_1(x, y) \not\equiv 0$  or  $Y_1(x, y) \not\equiv 0$ . For the case  $|\lambda_1| \neq 1$  or  $|\lambda_2| \neq 1$ , we obtained analytic general solutions of (1.2) in [5] and [6]. For a long time we could not treat the equation (1.2) under the condition  $|\lambda_1| = |\lambda_2| = 1$ , because it is difficult to have an analytic solution of the equation (1.2). For analytic solutions of a nonlinear first order difference equations, Kimura [1] and Yanagihara [7] studied the cases in which the absolute value of the eigenvalue equal to 1.

Next we consider a functional equation

$$\Psi(X(x, \Psi(x))) = Y(x, \Psi(x)), \quad (1.6)$$

where  $X(x, y)$  and  $Y(x, y)$  are holomorphic functions in  $|x| < \delta_1$ ,  $|y| < \delta_1$ . We assume that  $X(x, y)$  and  $Y(x, y)$  are expanded there as in (1.5).

As far as  $\frac{dx}{dt} \neq 0$ , an existence of solutions of (1.2) is equivalent to an existence of solution  $\Psi$  of (1.6). Furthermore we can reduce (1.2) to the following first order difference equation

$$x(t+1) = X(x(t), \Psi(x(t))), \quad (1.7)$$

Hereafter we consider  $t$  to be a complex variable, and concentrate on the difference system (1.2). Our aim in this paper is to show the following Theorem 1.

**Theorem 1** Suppose  $X(x, y)$  and  $Y(x, y)$  are expanded in the forms (1.5) such that  $X_1(x, y) \not\equiv 0$  or  $Y_1(x, y) \not\equiv 0$ .

(1) We define domains  $D_1(\kappa_0, R_0)$  by

$$D_1(\kappa_0, R_0) = \{t : |t| > R_0, |\arg[t]| < \kappa_0\}, \quad (1.8)$$

where  $\kappa_0$  is any constant such that  $0 < \kappa_0 \leq \frac{\pi}{4}$  and  $R_0$  is sufficiently large number which may depend on  $X$  and  $Y$ . Further define

$$D^*(\kappa, \delta) = \{x : |\arg[x]| < \kappa, 0 < |x| < \delta\}, \quad (1.9)$$

where  $\delta$  is a small constant and  $\kappa$  is a constant such that  $\kappa = 2\kappa_0$ , i.e.,  $0 < \kappa \leq \frac{\pi}{2}$ . Suppose that  $kc_{20} = d_{11} < 0$  for some  $k \in \mathbb{N}$ ,  $k \geq 2$ , and  $A = c_{20}$ , then we have a formal solution  $x(t)$  of (1.2) the following form

$$\frac{1}{At} \left( 1 + \sum_{j+k \geq 1} \hat{q}_{jk} t^{-j} \left( \frac{\log t}{t} \right)^k \right)^{-1}, \quad (1.10)$$

where  $\hat{q}_{jk}$  are constants which are defined by  $X$  and  $Y$ .

(2) Suppose  $R_1 = \max(R_0, 2/(|A|\delta))$ , then there is a solution  $x(t)$  of (1.2) such that  $x(t) \in D^*(\kappa, \delta)$  for  $t \in D_1(\kappa_0, R_1)$ , which the solution satisfying the following conditions:

(i)  $x(t)$  is holomorphic in  $D_1(\kappa_0, R_1)$ .

(ii)  $x(t)$  is expressible in the form

$$x(t) = -\frac{1}{At} \left( 1 + b\left(t, \frac{\log t}{t}\right) \right)^{-1}, \quad (1.11)$$

where  $b(t, \eta)$  is holomorphic for  $t \in D_1(\kappa_0, R_1)$ ,  $|\eta| < r$ , and in the expansion  $b(t, \eta) \sim \sum_{k=1}^{\infty} b_k(t) \eta^k$ ,  $b_k(t)$  is asymptotically develop-able into  $b_k(t) \sim \sum_{j+k \geq 1}^{\infty} b_{jk} t^{-j}$ , as  $t \rightarrow \infty$  through  $D_1(\kappa_0, R_1)$ , where  $b_{jk}$  are constants which are defined by  $X$  and  $Y$ .

## 2 Proof of Theorem 1

In [1], Kimura considered the following first order difference equation

$$w(t + \lambda) = F(w(t)), \quad (D1)$$

where  $F$  is represented in a neighborhood of  $\infty$  by a Laurent series

$$F(z) = z \left( 1 + \sum_{j=1}^{\infty} b_j z^{-j} \right), \quad b_1 = \lambda \neq 0. \quad (2.1)$$

He defined the following domains

$$D(\epsilon, R) = \{t : |t| > R, |\arg[t] - \theta| < \frac{\pi}{2} - \epsilon, \text{ or } \operatorname{Im}(e^{i(\theta-\epsilon)} t) > R, \\ \text{or } \operatorname{Im}(e^{i(\theta+\epsilon)} t) < -R\}, \quad (2.2)$$

$$\hat{D}(\epsilon, R) = \{t : |t| > R, |\arg[t] - \theta - \pi| < \frac{\pi}{2} - \epsilon \text{ or } \operatorname{Im}(e^{-i(\theta+\pi-\epsilon)}t) > R \\ \text{or } \operatorname{Im}(e^{-i(\theta+\pi+\epsilon)}t) < -R\}, \quad (2.3)$$

where  $\epsilon$  is an arbitrarily small positive number and  $R$  is a sufficiently large number which may depend on  $\epsilon$  and  $F$ ,  $\theta = \arg \lambda$ , (in this present paper, we consider the case  $\lambda = 1$  in (D1)). He proved the following theorems A and B.

**Theorem A.** Equation (D1) admits a formal solution of the form

$$t \left( 1 + \sum_{j+k \geq 1} \hat{q}_{jk} t^{-j} \left( \frac{\log t}{t} \right)^k \right) \quad (2.4)$$

containing an arbitrary constant, where  $\hat{q}_{jk}$  are constants defined by  $F$ .

**Theorem B.** Given a formal solution of the form (2.4) of (D1), there exists a unique solution  $w(t)$  satisfying the following conditions:

- (i)  $w(t)$  is holomorphic in  $D(\epsilon, R)$ ,
- (ii)  $w(t)$  is expressible in the form

$$w(t) = t \left( 1 + b \left( t, \frac{\log t}{t} \right) \right), \quad (2.5)$$

where the domain  $D(\epsilon, R)$  is defined by (2.2) and  $b(t, \eta)$  is holomorphic for  $t \in D(\epsilon, R)$ ,  $|\eta| < 1/R$ , and in the expansion  $b(t, \eta) \sim \sum_{k=1}^{\infty} b_k(t) \eta^k$ ,  $b_k(t)$  is asymptotically developable into  $b_k(t) \sim \sum_{j+k \geq 1}^{\infty} \hat{q}_{jk} t^{-j}$ , as  $t \rightarrow \infty$  through  $D(\epsilon, R)$ , where  $\hat{q}_{jk}$  are constants which are defined by  $X$  and  $Y$ .

Also there exists a unique solution  $\hat{w}$  which is holomorphic in  $\hat{D}(\epsilon, R)$  and satisfies a condition analogous to (ii), where the domain  $\hat{D}(\epsilon, R)$  is defined by (2.3).

In Theorem A and B, he defined the function  $F$  as in (2.1). But in our method, we can not have a Laurent series of the function  $F$ . Hence we derive following Propositions.

In the following,  $A$  denotes the constant  $A = c_{20}$  in Theorem 1, where  $c_{20}$  is the coefficient in (1.5).

**Proposition 2.** Suppose  $\tilde{F}(t)$  is holomorphic and expanded asymptotically in  $\{t; -1/(At) \in D^*(\kappa, \delta), A < 0\}$  as

$$\tilde{F}(t) \sim t \left( 1 + \sum_{j=1}^{\infty} b_j t^{-j} \right), \quad b_1 = \lambda \neq 0,$$

where  $D^*(\kappa, \delta)$  is defined in (1.9). Then the equation

$$\psi(\tilde{F}(t)) = \psi(t) + \lambda \quad (2.6)$$

has a formal solution

$$\psi(t) = t \left( 1 + \sum_{j=1}^{\infty} q_j t^{-j} + q \frac{\log t}{t} \right), \quad (2.7)$$

where  $q_1$  can be arbitrarily prescribed while other coefficients are uniquely determined by  $b_j$ , ( $j = 1, 2, \dots$ ), independently of  $q_1$ .

**Proposition 3.** *The equation (2.6) has a solution  $w = \psi(t)$ , which is holomorphic in  $\{t; -1/(At) \in D^*(\kappa/2, \delta/2), A < 0\}$  and has asymptotic expansion (2.7) there.*

These Propositions are proved as in [1] pp. 212–222. Since  $A = c_{20} < 0$  and  $\kappa_0 = \kappa/2$ , we see that  $x = -1/(At) \in D^*(\kappa/2, \delta/2)$  equivalent to  $t \in D_1(\kappa/2, 2/(|A|\delta)) = D_1(\kappa_0, 2/(|A|\delta))$ , where  $D_1(\kappa_0, R_0)$  is defined in (1.8). Further, as in [1] pp.206 and pp.228–232, we have following Proposition 4.

**Proposition 4.** *Suppose a function  $\phi$  is the inverse of  $\psi$  such that  $w = \psi^{-1}(t) = \phi(t)$ . Then we have  $\phi \circ \psi(w) = w, \psi \circ \phi(t) = t$ , furthermore  $\phi$  is holomorphic and asymptotically expanded in  $\{t; t \in D_1(\kappa_0, 2/(|A|\delta))\}$  as*

$$\phi(t) \sim t \left\{ 1 + \sum_{j+k \geq 1} \hat{q}_{jk} t^{-j} \left( \frac{\log t}{t} \right)^k \right\}. \quad (2.8)$$

This function  $\phi(t)$  is a solution of difference equation of (D1).

In [4], we proved the following theorem C.

**Theorem C.** *Suppose  $X(x, y)$  and  $Y(x, y)$  are defined in (1.5). Then*

(1) *if  $kc_{20} \neq d_{11}$  for any  $k \in \mathbb{N}, k \geq 2$ , then the formal solution  $\Psi(x)$  of (1.6) of the following form*

$$\Psi(x) = \sum_{m=1}^{\infty} a_m x^m, \quad (2.9)$$

*is identical to 0, i.e.,  $a_1 = a_2 = \dots = 0$ .*

(2) *if  $kc_{20} = d_{11}$  for some  $k \in \mathbb{N}, k \geq 2$ , then we have a formal solution  $\Psi(x)$  of (1.6) such the following form*

$$\Psi(x) = \sum_{m=k}^{\infty} a_m x^m, \quad (2.10)$$

*i.e.,  $a_1 = a_2 = \dots = a_{k-1} = 0$ .*

(3) *suppose*

$$kc_{20} = d_{11} < 0 \quad \text{for some } k \in \mathbb{N}, k \geq 2. \quad (2.11)$$

For any  $\kappa$ ,  $0 < \kappa \leq \frac{\pi}{2}$  and small  $\delta$ , we define the following domain  $D^*(\kappa, \delta)$ ,

$$D^*(\kappa, \delta) = \{x; |\arg[x]| < \kappa, 0 < |x| < \delta\}. \quad (1.10)$$

there is a constant  $\delta > 0$  and a solution  $\Psi(x)$  of (1.6), which is holomorphic and can be expanded asymptotically in  $D^*(\kappa, \delta)$  such that

$$\Psi(x) \sim \sum_{j=k}^{\infty} a_j x^j. \quad (2.12)$$

**Proof of Theorem 1.** We prove (1) of Theorem 1. We assume that  $kc_{20} = q_{11} < 0$  for some  $k \in \mathbb{N}$ , we suppose that  $R_0 > R$  and  $\kappa_0 < \frac{\pi}{4} - \epsilon$ . Since  $\theta = \arg[\lambda] = \arg[1] = 0$ , we have

$$D_1(\kappa_0, R_0) \subset D(\epsilon, R). \quad (2.13)$$

From Theorem C, for a  $x \in D^*(\kappa, \delta)$  we have a solution  $\Psi(x)$  of (1.6) which is holomorphic and can be expanded asymptotically in  $D^*(\kappa, \delta)$  such that

$$\Psi(x) \sim \sum_{j=k}^{\infty} a_j x^j. \quad (2.12)$$

On the other hand putting  $A = c_{20}$  and  $w(t) = -\frac{1}{Ax(t)}$  in (1.7), then we have

$$w(t+1) = -\frac{1}{AX\left(-\frac{1}{Aw(t)}, \Psi\left(-\frac{1}{Aw(t)}\right)\right)}. \quad (2.14)$$

If we can have  $-\frac{1}{Aw} = x \in D^*(\kappa, \delta)$ , then making use of Theorem C, we have a solution  $\Psi(x)$  of (1.6) such that  $\Psi(x) = \Psi\left(-\frac{1}{Aw}\right) \sim \sum_{m=k}^{\infty} a_j \left(-\frac{1}{Aw}\right)^m$ , ( $k \geq 2$ ). Further from (1.5), we have

$$-\frac{1}{AX\left(x, \Psi(x)\right)} \sim w \left[ 1 + c_{20} \frac{1}{A} w^{-1} + \sum_{k \geq 2} \tilde{c}_k(w)^{-k} \right], \quad (2.15)$$

where  $\tilde{c}_k$  are defined by  $c_{ij}$  and  $a_k$  ( $i+j \geq 2$ ,  $i \geq 1$ ,  $k \geq 2$ ). From (2.15) and definition of  $A$ , we can write (2.15) into the following form (2.16),

$$w(t+1) = \tilde{F}(w(t)) \sim w(t) \left\{ 1 + w(t)^{-1} + \sum_{k \geq 2} \tilde{c}_k(w(t))^{-k} \right\}. \quad (2.16)$$

On the other hand, putting  $\lambda = 1$  and  $m = 1$  in (2.1), i.e.  $\theta = 0$ , then making use of the Theorem A, we have the following first order difference equation ( $D1$ ,  $\lambda = 1$ )

$$w(t+1) = F(w(t)) = w(t) \left( 1 + w(t)^{-1} + \sum_{j=2}^{\infty} b_j w(t)^{-j} \right), \quad (D1, \lambda = 1)$$

admits a formal solution of the form  $t \left( 1 + \sum_{j+k \geq 1} \hat{q}_{jk} t^{-j} \left( \frac{\log t}{t} \right)^k \right)$ .

Similarly for the first order difference equation (2.16), making use of Proposition 2, we have a formal solution (2.17) of it such that,

$$w(t) = t \left( 1 + \sum_{j+k \geq 1} b_{jk} t^{-j} \left( \frac{\log t}{t} \right)^k \right), \quad (2.17)$$

where  $b_{jk}$  are defined by  $\tilde{F}$  in (2.16).

From  $x(t) = -\frac{1}{Aw(t)}$ , we have a formal solution of (1.2) such that

$$x(t) = -\frac{1}{At} \left( 1 + \sum_{j+k \geq 1} b_{jk} t^{-j} \left( \frac{\log t}{t} \right)^k \right)^{-1}. \quad (2.18)$$

Conversely, if we have a formal function  $x(t)$  such that in (2.18) exist in the domain  $D^*(\kappa, \delta)$ , then we can prove that the formal function (2.18) is a formal solution of (1.2), as  $t \rightarrow \infty$  through  $D_1(\kappa_0, R_0)$ . At first we take a small  $\delta > 0$ . For sufficiently large  $R$ , since  $R_0 > R$ , we can have

$$|x(t)| = \left| \frac{1}{At} \right| \left| 1 + \sum_{j+k \geq 1} b_{jk} t^{-j} \left( \frac{\log t}{t} \right)^k \right|^{-1} < \frac{1}{|A|R} (1+1) < \delta. \quad (2.19)$$

for  $t \in D_1(\kappa_0, R_0)$ . Since  $A = c_{20} < 0$ , if we take sufficiently large  $R_0$ , then we have

$$\left| \arg \left[ 1 + b \left( t, \frac{\log t}{t} \right) \right] \right| < \kappa_0, \quad \text{for } t \in D_i(\kappa_0, R_0).$$

Hence we have  $-\kappa_0 - \kappa_0 \leq \arg [x(t)] \leq \kappa_0 + \kappa_0$ . From the assumption of  $\kappa = 2\kappa_0$ , we have

$$|\arg [x(t)]| < \kappa \leq \frac{\pi}{2} \quad \text{for } t \in D_1(\kappa_0, R_0). \quad (2.20)$$

From (2.19) and (2.20), we have that  $x(t) \in D^*(\kappa, \delta)$  for a some  $\kappa$ ,  $(0 < \kappa \leq \frac{\pi}{2})$ .

Hence we have a  $\Psi(x(t))$  which satisfies the equation (1.6) and we prove that the function  $x(t)$  is a formal solution of (1.2) and holomorphic in  $D_1(\kappa_0, R_0)$ . Therefore we see that the function  $x(t)$  in the (2.18) is a formal solution of (1.2).

Next we prove (2). Suppose that  $R_1 = \max(R_0, 2/(|A|\delta))$ , making use of Proposition 4, then we have a holomorphic solution  $w(t)$  of (2.16) for  $t \in D_1(\kappa_0, R_1)$ , i.e., we have a solution  $x(t)$  of (1.2) for  $t$  at there, in which satisfying following conditions:

- (i)  $x(t)$  is holomorphic in  $D_1(\kappa_0, R_1)$ ,
- (ii)  $w(t)$  is expressible in the form

$$x(t) = -\frac{1}{At} \left( 1 + b \left( t, \frac{\log t}{t} \right) \right)^{-1}, \quad (2.21)$$



where  $b(t, \eta)$  is holomorphic for  $t \in D_1(\kappa_0, R_1)$ ,  $|\eta| < r$ , and in the expansion  $b(t, \eta) \sim \sum_{k=1}^{\infty} b_k(t) \eta^k$ ,  $b_k(t)$  is asymptotically develop-able into  $b_k(t) \sim \sum_{j+k \geq 1}^{\infty} b_{jk} t^{-j}$ , as  $t \rightarrow \infty$  though  $D_1(\kappa_0, R_1)$ .  $\square$

Finally, we have a solution  $u(t), v(t)$  of (1.1) by the transformation

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = P \begin{pmatrix} x(t) \\ \Psi(x(t)) \end{pmatrix}.$$

## References

- [1] T. Kimura, *On the Iteration of Analytic Functions*, Funkcialaj Ekvacioj, **14**, (1971), 197-238.
- [2] M. Suzuki, *Holomorphic solutions of some functional equations*, Nihonkai Math. J., **5**, (1994), 109-114.
- [3] M. Suzuki, *Holomorphic solutions of some system of  $n$  functional equations with  $n$  variables related to difference systems*, Aequationes Mathematicae, **57**, (1999), 21-36.
- [4] M. Suzuki, *Holomorphic solutions of some functional equations II*, Southeast Asian Bulletin of Mathematics, **24**, (2000), 85-94.
- [5] M. Suzuki, *Analytic General Solutions of Nonlinear Difference Equations*, preprint.
- [6] M. Suzuki, *Holomorphic solutions of a Functional Equation*, preprint.
- [7] N. Yanagihara, *Meromorphic solutions of some difference equations*, Funkcialaj Ekvacioj, **23**, (1980), 309-326.